An integrable q-deformed model for bosons interacting with spin impurities

1994 J. Phys. A: Math. Gen. 27 L363

(http://iopscience.iop.org/0305-4470/27/11/003)

J. Phys. A: Math. Gen. 27 (1994) L363–L367. Printed in the UK

LETTER TO THE EDITOR

An integrable q-deformed model for bosons interacting with spin impurities

N M Bogoliubov†‡, A V Rybin† and J Timonen†

† Department of Physics, University of Jyväskylä, PO Box 35, FIN-40351 Jyväskylä, Finland
‡ Research Institute for Theoretical Physics, PO Box 9, FIN-00014, University of Helsinki, Finland

Received 4 March 1994

Abstract. A composite quantum model on a lattice which describes the system of q-bosons interacting with $U_q[su(2)]$ spin impurities is introduced and solved exactly under periodic boundary conditions. In one limit the model is shown to become a new exactly soluble quantum system on a lattice which can be interpreted as a q-deformed version of the quantum Dicke model. In the limit of infinitesimal spacing the model is further reduced to a multilevel version of the previously introduced continuum-limit Dicke model. For spin $\frac{1}{2}$ the previous results for this particular case are confirmed.

It has recently become obvious that integrable strongly interacting models can be often expressed in a particularly simple form in terms of q-deformed variables. Thus far considerations have been mostly restricted to models with only one type of particle. We shall show here that similar methods can be applied to models with several types of particles, and in this letter we shall report an exact solution of a quantum model on a lattice which describes a system of q-bosons interacting with $U_q[su(2)]$ spin impurities. This model is defined by the Hamiltonian

$$\mathcal{H}_D = \frac{1}{2}\sum_{n=1}^{M} J(q)\Big[\chi_n^+(\beta_n+\beta_{n+1}) + \chi_n^-(\beta_n^\dagger+\beta_{n+1}^\dagger)\Big] - \mathrm{i}q^{2H_n}(\beta_{n+1}^\dagger\beta_n - \beta_n^\dagger\beta_{n+1}) \qquad (1)$$

with $J(q) = \sqrt{q-q^{-1}}$, and we shall choose here $q = \mathrm{e}^\gamma > 1$. We shall also impose periodic boundary conditions $n+M \equiv n$ (the number of lattice sites is even for simplicity). The annihilation and creation operators β_n, $\beta_n^\dagger$ in (1), together with the related number operator $N_n = N_n^\dagger$, form a q-boson algebra with the commutation relations

$$[\beta_n, \beta_m^\dagger] = q^{2N_n}\delta_{mn} \qquad [N_n, \beta_m^\dagger] = \beta_n^\dagger\delta_{mn} \qquad \beta_n = (\beta_n^\dagger)^\dagger . \qquad (2)$$

The q-spin operators in (1) are defined as $\chi_n^- = q^{H_n}X_n^-$, $\chi_n^+ = X_n^+q^{H_n}$, where $X_n^\pm$ and H_n are quantum operators belonging to the irreducible $(2S+1)$-dimensional representation of the quantum $U_q[su(2)]$ algebra. They satisfy

$$[X_n^+, X_m^-] = [2H_n]\delta_{mn} \qquad [H_n, X_m^\pm] = \pm X_n^\pm\delta_{mn} . \qquad (3)$$

Here $[\cdot,\cdot]$ denotes the usual commutator, while the 'box' notation $[\cdot]$ means the operation $[x] = (q^x - q^{-x})/(q-q^{-1})$.

The q-boson operators in (1) can be expressed [1] in terms of the ordinary boson operators $\varepsilon_n, \varepsilon_n^\dagger$; $[\varepsilon_n, \varepsilon_m^\dagger] = \delta_{nm}$, such that $\beta_n = (\beta_n^\dagger)^\dagger = q^{N_n}\varepsilon_n\,([N_n]/N_n)^{1/2}$, and $N_n = \varepsilon_n^\dagger\varepsilon_n$.

The q-spin operators may be realized in terms of the usual $su(2)$ generators S_n^3 and $S_m^\pm$; $[S_n^3, S_m^\pm] = \pm S_n^\pm \delta_{nm}$, $[S_n^+, S_m^-] = 2S_n^3\delta_{nm}$, in the form $X_n^+ = (X_n^-)^\dagger = S_n^+\left([S_n^3 - S][S_n^3 + S + 1]/(S_n^3 - S)(S_n^3 + S + 1)\right)^{1/2}$, $H_n = S_n^3$. These representations reflect the physical meaning of the introduced model. We can consider the model with Hamiltonian (1) as one of strongly correlated *ordinary* bosons interacting with *ordinary* spin impurities. The exchange energy of the system depends on the occupation of lattice sites (correlated hopping) and on the the value of the third spin component.

Since the Pauli matrices σ^3 and $\sigma^\pm$ satisfy the q-spin commutation relations (3), the Hamiltonian (1) may also be used to describe a chain of spins ($S = \frac{1}{2}$) that interact via a q-boson field. When the deformation parameter $q \to 1$, the q-deformed algebras become the conventional ones, and the model (1) becomes the linear free-boson model.

In the continuum limit as the lattice spacing $\Delta \to 0$, we should distinguish between two cases: that of a 'discrete' and that of a 'continuous' medium. In the limit

$$\Delta \to 0 \qquad L = M\Delta \qquad x = n\Delta \qquad \beta_n = \sqrt{\Delta}\varepsilon(x) \qquad \gamma = \kappa\Delta/2 \qquad \kappa > 0 \tag{4}$$

we have $\chi_n^\pm = S_n^\pm + \mathcal{O}(\Delta)$, $H_n = S_n^3$, $[\varepsilon(x), \varepsilon^\dagger(y)] = \delta(x-y)$, and the Hamiltonian of the continuum-limit Dicke model with discrete atoms of spin S is recovered:

$$H_{\mathrm{D}} = +\frac{\mathrm{i}}{2}\int_0^L \mathrm{d}x\,\left[\left(\partial_x\varepsilon^\dagger(x)\right)\varepsilon(x) - \varepsilon^\dagger(x)\partial_x\varepsilon(x)\right] - \sqrt{\kappa}\sum_{n=1}^{M}(S_n^+\varepsilon(x_n) + S_n^-\varepsilon^\dagger(x_n)). \tag{5}$$

For $S = \frac{1}{2}$ this is the two-level Dicke model for a 'discrete' medium considered in [2]. The corresponding 'continuous' case of the Dicke model is obtained by setting $S_n^{\pm,3} = \Delta S^{\pm,3}(x)$ with $\left[S^+(x), S^-(y)\right] = 2S^3(x)\delta(x-y)$, and $\left[S^3(x), S^\pm(y)\right] = \pm S^\pm(x)\delta(x-y)$. The established correspondence means that the model defined by Hamiltonian (1) may be considered as the *q-deformation* of the multi-level Dicke model.

In order to solve the models (1) and (5) we construct first a generic composite quantum model which we solve for its eigenstates and eigenvalues by the quantum inverse-scattering method (QISM) [3]. Notice that composite models have previously been used in the theory of spin chains [4]. We define the generic model by the ℓ-operator

$$\ell(\lambda|n) = L^{\mathrm{S}}(\lambda|n)L^{\mathrm{B}}(\lambda|n) \tag{6}$$

where the q-spin L-operator L^{S} is given by

$$L^{\mathrm{S}}(\lambda|n) = \begin{pmatrix} [\mathrm{i}\lambda - H_n] & -X_n^+ \\ X_n^- & -[\mathrm{i}\lambda + H_n] \end{pmatrix}. \tag{7}$$

This is also the L-operator of the XXZ chain of arbitrary spin [5].

The q-boson operator $L^{\mathrm{B}}(\lambda|n)$ can be obtained from $L^{\mathrm{S}}(\lambda|n)$ through application of an appropriate Holstein–Primakov transformation [6]:

$$L^{\mathrm{B}}(\lambda|n) = \begin{pmatrix} [\mathrm{i}\lambda + \frac{1}{2}\Lambda_n - N_n] & \mathrm{i}\beta_n^\dagger q^{-\frac{1}{2}N_n}\sqrt{[\Lambda_n - N_n]} \\ -\mathrm{i}\sqrt{[\Lambda_n - N_n]}q^{-\frac{1}{2}N_n}\beta_n & [\mathrm{i}\lambda - \frac{1}{2}\Lambda_n + N_n] \end{pmatrix}. \tag{8}$$

Here and in (7) $\lambda \in \mathbb{C}$ is a spectral parameter. We shall also choose in the following $\Lambda_n = (-1)^n + \Lambda$, $\Lambda \in \mathbb{R}$.

The ℓ-operator (6) satisfies the bilinear intertwining relation

$$R(\lambda, \mu)\ell(\lambda|n) \otimes \ell(\mu|n) = \ell(\mu|n) \otimes \ell(\lambda|n)R(\lambda - \mu) \tag{9}$$

with the trigonometric R-matrix

$$R(\lambda-\mu)=\begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda) \end{pmatrix} \tag{10}$$

where $f(\mu,\lambda)=[\mathrm{i}(\lambda-\mu)+1]/[\mathrm{i}(\lambda-\mu)]$ and $g(\mu,\lambda)=1/[\mathrm{i}(\lambda-\mu)]$.

It also satisfies the involution property

$$\ell(\lambda|n)=-\sigma^2\ell^{\dagger}(n|\bar{\lambda})\sigma^2 \tag{11}$$

in which σ^2 is a Pauli matrix and the Hermitian conjugation is taken *only* with respect to the quantum space. The quantum determinant [3] of the ℓ-operator (6) is given by

$$\begin{aligned}\hat{I}\cdot \text{q-det}\ell(\lambda|n)&=\ell(\lambda|n)\sigma^2\ell(\lambda+\mathrm{i}|n)\sigma^2\\ &=\hat{I}\cdot\text{q-det}L^S(\lambda|n)\text{q-det}L^{\mathrm{B}}(\lambda|n)\,.\end{aligned} \tag{12}$$

From this we find immediately that q-det$L^{\mathrm{B}}(\lambda|n)=[\mathrm{i}\lambda+\frac{1}{2}\Lambda_n][\mathrm{i}\lambda-\frac{1}{2}\Lambda_n-1]$.

The transfer matrix $t(\lambda)$, which is the generating function of the integrals of motion, and the monodromy matrix $T(\lambda)$, are introduced in the usual way such that $t(\lambda)=\mathrm{tr}T(\lambda)=A(\lambda)+D(\lambda)$, where

$$\begin{aligned}T(\lambda)&=\ell(\lambda|M)\ell(\lambda|M-1)\ldots\ell(\lambda|1)\\ &=\begin{pmatrix} A(\lambda) & B(\lambda)\\ C(\lambda) & D(\lambda)\end{pmatrix}.\end{aligned} \tag{13}$$

The property $[t(\lambda),t(\mu)]=0$ can easily be deduced from the bilinear intertwining relation (9).

From (11) it follows further that

$$t^{\dagger}(\lambda)=t(\bar{\lambda}) \qquad \left[t(\lambda),t^{\dagger}(\lambda)\right]=0\,. \tag{14}$$

The Hamiltonian of the composite model is constructed by means of the projector method [3, 6]. This method is based on the fact that, at the points where the quantum determinant (12) vanishes, one can factorize the generating function of the integrals of motion, $t(\lambda)$. The introduced dependence through Λ_n of the L-operator $L^{\mathrm{B}}(\lambda|n)$ on the parity of the lattice site n, removes the problems associated with the non-commutativity of the matrix elements of $L^{\mathrm{B}}(\lambda|n)$.

At the point $\lambda=\nu=-\frac{1}{2}\mathrm{i}(\Lambda+1)$, where q-det$L^{\mathrm{B}}(\lambda|n)=0$, the operator $L^{\mathrm{B}}(\lambda|n)$ has the structure

$$L^{\mathrm{B}}(\nu|2k+1)_{ij}=g_i(2k+1)g_j^{\dagger}(2k+1) \qquad L^{\mathrm{B}}(\nu|2k)_{ij}=g_j^{\dagger}(2k)g_i(2k) \tag{15}$$

where $g_1(n)=\sqrt{[\Lambda-N_n+\epsilon_{n+1}]}$, $g_2(n)=-\mathrm{i}q^{-\frac{1}{2}N_n}\beta_n$, and $\epsilon_n=\frac{1}{2}(1-(-1)^n)$.

After some algebra we find for $\lambda=\nu$ that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\ln t(\lambda)\big|_{\lambda=\nu}=\sum_{n=1}^{M}t_n \tag{16}$$

where the local densities t_n are given by

$$\begin{aligned}t_n=\mathrm{i}\langle g^{\dagger}(n-1+\epsilon_n),\,&L^{\mathrm{S}}(\nu|n-2+\epsilon_n)g(n-2+\epsilon_n)\rangle^{-1}\\ &\times\langle g^{\dagger}(n+1+\epsilon_n),L^{\mathrm{S}}(\nu|n+\epsilon_n)g(n+\epsilon_n)\rangle^{-1}\\ &\times\langle g^{\dagger}(n+\epsilon_n),L^{\mathrm{S}}(\nu|n-1+\epsilon_n)g(n-1+\epsilon_n)\rangle^{-1}\\ &\times\langle g^{\dagger}(n+1),G(\nu|n)L^{\mathrm{S}}(\nu|n-1)g(n-1)\rangle\\ &\times\langle g^{\dagger}(n-1+3\epsilon_n),L^{\mathrm{S}}(\nu|n-2+3\epsilon_n)g(n-2+3\epsilon_n)\rangle\,.\end{aligned} \tag{17}$$

In these expressions $G(\lambda|n) = \mathrm{d}\ell(\lambda|n)/\mathrm{d}\lambda$, $\langle\cdot,\cdot\rangle$ stands for the usual scalar product $\langle a, b\rangle = a_1b_1 + a_2b_2$, and $g(n)$ is the operator-valued vector $(g_1(n), g_2(n))^T$.

We shall make here a particular choice for the Hamiltonian, namely

$$\mathcal{H} = \frac{1}{4(q-q^{-1})\ln q}\sum_{n=1}^{M}\left(t_n + t_n^{\dagger}\right) . \tag{18}$$

It follows immediately from (14) and (18) that $[\mathcal{H}, t(\lambda)] = 0$. We can now apply the method of algebraic Bethe ansatz [3] to find the spectrum of the generic Hamiltonian (18). The quantum space of the model is the tensor product of the Fock space and the space of the representation of the $U_q[su(2)]$ algebra. The local vacuum vector on each lattice site n is given by $|0\rangle_n = |S, -S\rangle_n^{\mathrm{S}} \otimes |0\rangle_n^{\mathrm{B}}$, where $|S, -S\rangle_n^{\mathrm{S}}$ is the lowest vector of the representation of $U_q[su(2)]$, and $|0\rangle_n^{\mathrm{B}}$ is the vacuum vector of the Fock space. The global vacuum vector of the model is evidently $|0\rangle = \bigotimes_{n=1}^{M}|0\rangle_n$.

In the framework of the algebraic Bethe-ansatz method the operator $B(\lambda)$ (cf equation (13)) plays the role of a creation operator of a quasi-particle excitation, and hence the N-particle eigenfunction of the transfer matrix can be constructed as

$$|\psi_N(\{\lambda_j\})\rangle = \prod_{j=1}^{N} B(\lambda_j)|0\rangle \tag{19}$$

where the parameters $\{\lambda_j\}$ satisfy the Bethe equations

$$\left(\frac{a(\lambda_l)}{d(\lambda_l)}\right)^{M/2} = \prod_{\substack{j=1\\ j\neq l}}^{N}\frac{[\mathrm{i}\lambda_l - \mathrm{i}\lambda_j + 1]}{[\mathrm{i}\lambda_l - \mathrm{i}\lambda_j - 1]} \tag{20}$$

with

$$\begin{aligned} a(\lambda) &= [\mathrm{i}\lambda + \tfrac{1}{2}(\Lambda - 1)][\mathrm{i}\lambda + \tfrac{1}{2}(\Lambda + 1)][\mathrm{i}\lambda + S]^2 \\ d(\lambda) &= [\mathrm{i}\lambda - \tfrac{1}{2}(\Lambda - 1)][\mathrm{i}\lambda - \tfrac{1}{2}(\Lambda + 1)][i\lambda - S]^2 . \end{aligned} \tag{21}$$

The eigenvalues of $t(\mu)$ corresponding to these eigenfunctions are

$$\theta_N(\mu, \{\lambda_j\}) = a^{M/2}(\mu)\prod_{j=1}^{N} f(\mu, \lambda_j) + d^{M/2}(\mu)\prod_{j=1}^{N} f(\lambda_j, \mu) \tag{22}$$

where $f(\mu, \lambda)$ are the elements of the R-matrix (10). Using the definition (16) we finally find from (22) that the eigenvalues of the Hamiltonian (18) for an N-particle state are given as

$$\mathcal{H}|\psi_N(\{\lambda_j\})\rangle = \sum_{k=1}^{N} h(\lambda_k)|\psi_N(\{\lambda_j\})\rangle \tag{23}$$

where

$$h(\lambda_k) = \frac{\mathrm{i}}{2(q-q^{-1})^2}\left(\frac{1}{[\mathrm{i}\lambda_k - \frac{1}{2}(\Lambda+1)][\mathrm{i}\lambda_k - \frac{1}{2}(\Lambda-1)]} - \frac{1}{[\mathrm{i}\lambda_k + \frac{1}{2}(\Lambda+1)][\mathrm{i}\lambda_k + \frac{1}{2}(\Lambda-1)]}\right). \tag{24}$$

To establish the connection between (1) and the generic model (18), we shall use the procedure introduced in [7]. Noticing that Λ is a free real parameter of the generic model, we can study the limit $\Lambda \to \infty$. By direct calculation one can prove that $\mathcal{H}_{\mathrm{D}} = \lim_{\Lambda\to\infty} q^{\Lambda}\mathcal{H}$. The integrability of the model (1) is then guaranteed because the R-matrix (10) does not depend on Λ. As can be easily seen, for $\Lambda \to \infty$, $B(\lambda) = q^{M\Lambda/2}\big(B_0(\lambda) + \mathcal{O}(q^{-\Lambda})\big)$, and the eigenstates of $\mathcal{H}_D$ are $|\psi_N^{(0)}(\{\lambda_j\})\rangle = \prod_{j=1}^{N} B_0(\lambda_j)|0\rangle$.

The N-particle energy of the Hamiltonian (1) is then given by

$$E_N = -\sum_{k=1}^{N} \sin 2\gamma\lambda_k \tag{25}$$

which is parametrized by $\{\lambda_j\}$ that now satisfy their respective Bethe equations

$$\mathrm{e}^{-\mathrm{i}p(\lambda_l)M} = \prod_{\substack{j=1\\ j\neq l}}^{N} \frac{\sin\gamma(\lambda_l - \lambda_j - \mathrm{i})}{\sin\gamma(\lambda_l - \lambda_j + \mathrm{i})}. \tag{26}$$

Here $p(\lambda)$ is the momentum of a quasiparticle:

$$p(\lambda) = \mathrm{i}\ln \mathrm{e}^{2\mathrm{i}\gamma\lambda}\frac{\sin\gamma(\lambda - \mathrm{i}S)}{\sin\gamma(\lambda + \mathrm{i}S)}. \tag{27}$$

These equations possess solutions with both real $\lambda_l \in \mathbb{R}$ and complex $\lambda_l^{(n)} = \alpha_l - \frac{1}{2}\mathrm{i}(n + 1 - 2m) + \mathcal{O}\left(\mathrm{e}^{-M}\right)$; $m = 1, 2, \ldots, n$; $\operatorname{Im}\alpha_l = 0$.

In the 'discrete' medium limit (4) the Bethe equations (26) (up to the replacement of λ by λ/κ) take the form

$$\mathrm{e}^{\mathrm{i}\lambda_l L}\left(\frac{\lambda_l - \mathrm{i}S\kappa}{\lambda_l + \mathrm{i}S\kappa}\right)^M = \prod_{\substack{j=1\\ j\neq l}}^{N} \frac{\lambda_l - \lambda_j - \mathrm{i}\kappa}{\lambda_l - \lambda_j + \mathrm{i}\kappa}. \tag{28}$$

The N-particle energy of the Hamiltonian (5) is given by $E_N = -\sum_{k=1}^{N}\lambda_k$. In the particular case of $S = \frac{1}{2}$ the respective results of [2] are recovered. Replacing S by $S\Delta$, we further arrive at a continuum-limit Dicke model with continuously distrubuted atoms [8].

In this letter we have introduced, and solved exactly, a model of correlated lattice bosons interacting strongly with spin impurities. The Hamiltonian of the model takes a simple form in terms of q-deformed boson and spin variables, and this new q-deformed model may be regarded as a lattice version of the multi-level Dicke model. In the continuum limit this model is shown to reduce to a multi-level generalization of the previously studied Dicke model with discrete atoms. The detailed study of the spectra of the introduced models and their correlation functions will be given elsewhere. It is worth mentioning, however, that the existence in our model of complex-valued solutions (bound states), indicates the existence of the superradiance effect [2, 8].

AV Rybin thanks Professor H Paul and Dr H Steudel for the kind hospitality extended to him during his stay at the Max Planck Society while this work was in progress. Illuminating discussions with Dr H Steudel are also gratefully acknowledged.

References

[1] Chaichian M, Ellinas D and Kulish P P 1990 *Phys. Rev. Lett.* **65** 980
[2] Rupasov V I and Yudson V I 1984 *Sov. Phys.–JETP* **59** 478
[3] Korepin V E, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions Algebraic Bethe Ansatz* (Cambridge: Cambridge University Press)
[4] de Vega H J and Woynarovich F 1992 *J. Phys. A: Math. Gen.* **25** 4499
[5] Kulish P P and Reshetikhin N Yu 1983 *J. Sov. Math.* **23** 2435
[6] Bogoliubov N M and Bullough R K 1992 *J. Phys. A: Math. Gen.* **25** 4057
[7] Bogoliubov N M, Bullough R K and Pang G D 1993 *Phys. Rev.* B **47** 11 495
[8] Rupasov V I 1982 *Sov. Phys.–JETP* **56**